International Journal of Theoretical Physics, Vol. 37, No. 5, 1998

Dynamical Trajectories of Simple Mechanical Systems as Geodesics in Space with an Extra Dimension

Marek Szydlowski,[1] Andrzej J. Maciejewski,[2] and Jacek Guzik[1]

Received July 31, 1997

We show the advantages of representing the dynamics of simple mechanical systems described by a natural Lagrangian, in terms of geodesics of a Riemannian (or pseudo-Riemannian) space with an additional dimension. We demonstrate how general trajectories of simple mechanical systems can be put into one-to-one correspondence with the geodesics of a suitable manifold. Two different ways in which the geometry of the configuration space can be obtained from a higher dimensional model are presented and compared: (1) by a straightforward projection, and (2) as a space geometry of a quotient space obtained by the action of the timelike Killing vector generating a stationary symmetry of a background space geometry with an additional dimension. The second model is more informative and coincides with the so-called optical model of the line-of-sight geometry. On the base of this model we study the behavior of nearby geodesics to detect their sensitive dependence on initial conditions—the key ingredient of deterministic chaos. The advantage of such a formulation is its invariant character.

1. DYNAMICAL TRAJECTORIES OF SIMPLE MECHANICAL SYSTEMS AS GEODESICS IN SPACE WITH AN EXTRA DIMENSION

Let us consider a simple mechanical system, i.e., a system described by the natural Lagrange function

[1] Astronomical Observatory, Jagiellonian University, Orla 171, 30244 Krakow, Poland; e-mail: szydlo@oa.uj.edu.pl, guzik@oa.uj.edu.pl.
[2] Institute of Astronomy, N. Copernicus University, 87-100 Toruń, Chopina 12/18, Poland; e-mail: maciejka@astri.uni.torun.pl.

0020-7748/98/0500-156 9$15.00/0

$$\mathscr{L} = \frac{1}{2} g_{ij}\dot{q}^i\dot{q}^j - V(q), \qquad i, j = 1, \ldots, n \tag{1}$$

where q^i are local coordinates on the configuration space $\mathcal{M}$ that carries a metric $g = [g_{ij}]$ defined by the kinetic energy[3] $T = 1/2 g_{ij}\dot{q}^i\dot{q}^j; V(q)$ is a C^∞ real function of coordinates q^i. We assume that the Lagrange function (1) does not depend on the time variable t (also called mechanical time). Systems (1) with a positive- (or negative-) definite metric g will be called classical mechanical systems (Szydlowski, 1994; Szydlowski *et al.*, 1996), whereas those with an indefinite metric will be called relativistic mechanical systems (Szydlowski, 1997).

Table I illustrates that, in applications to general relativity and cosmology, we need simple relativistic systems (SRS) where the kinetic energy form is indefinite with the Lorentz signature.

The theory of SRS is still *in statu nascendi*, and the situation in this field is analogous to that which once compelled mathematicians to investigate spaces with Lorentz metrics.

In the generic situation all models from Table I exhibit a complex behavior of trajectories in phase space. In the class of homogeneous cosmological models there is an important class of Mixmaster models (Bianchi IX for $n_1 = n_2 = n_3 = 1$ and Bianchi VIII for $n_1 = n_2 = -n_3 = 1$) with complex behavior. Moreover, there are numerical and analytical arguments which confirm that this type of behavior is typical for the very early stage of the universe. Unfortunately, it is difficult to decide whether the Bianchi IX model is chaotic in the precise sense (Szydlowski and Krawiec, 1996; Szydlowski and Szczesny, 1994). The problem is strictly connected with an adequate understanding of chaos in the special contexts of general relativity and cosmology.

The Hamiltonian function for system (1) is of the form

$$\mathscr{H}(p, q) = \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta + V(q) \tag{2}$$

$$p_\alpha = g_{\alpha\beta}\dot{q}^\beta \tag{3}$$

General relativity applications allow us to consider only the zero energy level, i.e.,

$$\mathscr{H} = 0 \Leftrightarrow g_{ij}\dot{q}^i\dot{q}^j = -2V(q) \tag{4}$$

Therefore, trajectories of the system in TR^N with the coordinates $(q^\alpha, \dot{q}^\alpha)$ are situated in the domain given by

[3] Throughout this paper we use the convention that a repeated index indicates summation unless stated otherwise. Latin indices take values 1 to n.

Table I. Examples of Simple Relativistic System Whose Trajectories in Phase Space Show Complex (Chaotic) Behavior

Mechanical system	Hamilton (Lagrange) function	Remarks
Friedmann–Robertson–Walker cosmology coupled to real, free massive scalar field	$\mathcal{H} = \frac{1}{2}(-p_1^2 + p_2^2)$ $+\frac{1}{2}(-q_1^2 + q_2^2 + m^2 q_1^2 q_2^2)$ $= 0$	$m =$ const (Calzetta and El Hasi, 1993)
Friedmann–Robertson–Walker model with conformally coupled massive, real self-interacting scalar field	$\mathcal{H} = \frac{1}{2}(-p_a^2/m_p^2 + p_\phi^2)$ $+\frac{1}{2}\phi^2(m^2 V(a) + k^2)$	$m, m_p, k =$ const (Blanco *et al.*, 1995; Calzetta and Gonzales, 1995)
Friedmann–Robertson–Walker model with squared terms in action	$\mathcal{H} = \frac{1}{4}(p_1^2 - p_2^2) + \frac{1}{4}(q_1^2 - q_2^2)$ $+ (q_1^2/8\beta)(-q_1 + q_2)^2$	$q_1 + q_2 \geq 0$, $\beta =$ const (Hawking and Luttrell, 1984)
Bianchi IX cosmological model as a perturbed Toda lattice system	$\mathcal{H} = 2\, \Sigma_{i<j}^3\, p_i p_j - \Sigma_i\, p_i^2$ $+ 2\, \Sigma_{i<j}^3 \exp(q_i + q_j)$ $- \Sigma_{i=1}^3 \exp(2q_i)$	(Bogoyavlenski, 1976; Biesiada and Szydlowski, 1991)
Multidimensional generalization of models of Bianchi class $A \times B^{\mathrm{D}}$, where B^{D} is D- dimensional compact, homogenous space, $D = n - 3$	$\mathcal{H} = \frac{2}{n-1} \Pi_{i=1}^n q_i \{2\Sigma_{i<j}^n p_i p_j q_i q_j$ $-(n-2)\, \Sigma_{i=1}^n p_i^2 q_i^2\}$ $+\frac{1}{4}(\Pi_{i=1}^n q_i)^{1+\gamma}\{(q_1 q_2 q_3)^{-1}$ $\times [2\, \Sigma_{i<j}^n n_i n_j q_i q_j$ $-\Sigma_{i=1}^n n_i^2 n_i^2] + \ldots\}$	$n_i = 0, \pm 1$ (Szydlowski and Pajdosz, 1989)
Bianchi cosmological models with the ideal fluid with the equation of state $p = (\gamma - 1)\rho$	$\mathcal{H} = \frac{1}{(q_1 q_2 q_3)^{(1-\gamma)/2}} (T(p_i q_i)$ $+ \frac{1}{4} V(q_i))$ $T(p_i q_i) = 2\, \Sigma_{i<j}^3\, p_i p_j q_i q_j$ $- \Sigma_{i=1}^3\, p_i^2 q_i^2$ $V(q_i) = 2\, \Sigma_{i<j}^3\, n_i n_j q_i q_j$ $- \Sigma_{i=1}^3\, p_i^2 q_i^2$	$q_1 \propto a_i^2$, where a_i is a three scale factor in different main directions; $\gamma =$ const; $n_i = 0, \pm 1$ for different Bianchi models (Bogoyavlenski and Novikov, 1973)

Table I. Continued

Mechanical system	Hamilton (Lagrange) function	Remarks
Friedmann–Robertson–Walker model with non-minimally coupled scalar field	$\mathcal{H} = N\left\{-\frac{1}{4a^2}p_a^2 + \frac{1}{4a_{3-12\xi}}p_\xi^2 - U(a,\phi)\right\}$ $U(a,\xi) = a - g^2a^3 - \pi^2 a^3 V\left(\frac{\xi}{\pi a^{6\xi}}\right) - 6\xi a^{1-12\xi}\xi^2$	g, π = const; ϕ is a scalar field, $V(\phi)$ is the potential of a scalar field, a is a scale factor, ξ is a coupling constant (Feinberg and Peleg, 1995)
Charged particle in uniform magnetic field and linearly polarized gravitational wave	$\mathcal{L} = \frac{1}{2}\pi_0^2 - \frac{1}{2}\pi_1^2 - \frac{1}{2}\frac{(g_t)^2}{1-\alpha\sin[\nu(x^2-x^0)]} \equiv \frac{1}{2}$	α is the amplitude of a wave, ω is the angular frequency of a wave, Ω is the Larmour angular frequency of a charged particle, $\nu = \varphi/\Omega$
General relativity with the scalar field in ADM canonical formulation	$\mathcal{H} = \frac{1}{2}G^{AB}\Pi^A\Pi^B - \sqrt{g}^3R + \frac{\sqrt{g}}{2}\left[\frac{\Pi_\phi^2}{s}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right]$	g = det $g_{\alpha\beta}$ (de Witt, 1967; Misner *et al.*, 1973; Arnowitt *et al.*, 1962)
The motion of a test of particle or photon in space-time $\mathcal{M}$ of general relativity	$\mathcal{H} = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu = -m^2$	μ, ν = 0, 1, 2, 3; m is the mass parameter(m = 0 for photons), ds^2 is the metric of spacetime with Lorentzian signature
Cosmological models with the complex scalar self-interacting inflation field nonminimally coupled to gravity	$\mathcal{L} = \frac{m_p^2}{16\pi}(6a(-\dot{a}^2+1) - \Lambda a^3) + 3\xi a(-\dot{a}^2+1)x^2 - 6\xi\dot{a}\dot{x}a^2x + \frac{1}{2}\dot{x}^2a^3 - \frac{a^2}{2a^3z^2} - \frac{1}{2}m^2x^2a^3 - \frac{1}{41}\lambda x^4a^3$	(Kamenshchik *et al.*, 1996)

$$\Omega = \{(q^\alpha, \dot{q}^\alpha) \in R^{2N}: g_{ij}\dot{q}^i\dot{q}^j = -2V(q)\}$$

The Lagrange–Euler equations with Lagrangian (1) have the form

$$g_{ij}\ddot{q}^j + [jk, i]\dot{q}^j\dot{q}^k + \frac{\partial V}{\partial q^i} = 0, \tag{5}$$

where $[jk, i]$ is the Christoffel symbol of the first kind, namely

$$[jk, i] = \frac{1}{2}\left(\frac{\partial g_{ji}}{\partial q^k} + \frac{\partial g_{ki}}{\partial q^j} - \frac{\partial g_{ik}}{\partial q^i}\right)$$

These equations admit $E = T + V(q)$, as a first integral.

Following Eisenhart (1929), the trajectories of simple mechanical systems correspond to the geodesics of a suitable higher dimensional space endowed with the metric

$$ds^2 = g_{ij}\, dq^i\, dq^j + A(q^k)\, du^2 \equiv g_{\alpha\beta}\, dq^\alpha\, dq^\beta; \qquad \alpha, \beta = 1, \ldots, n + 1 \tag{6}$$

where we denote $q^{n+1} = u$. The correspondence mentioned above is one-to-one, i.e., expressions for trajectories are the same as those for geodesics of the metric (6). It is assumed that A does not depend on u. The requirements that guarantee the correspondence are the following:

$$t = as \tag{7}$$

$$\frac{1}{2A} = V + b \tag{8}$$

$$\frac{1}{a^2} = g_{ij}\dot{q}^i\dot{q}^j + \frac{1}{A} = 2(E + b) \tag{9}$$

where $a \neq 0$ for nonnull geodesics, and b is a constant which should be chosen consistently with the Hamiltonian constraint $\mathcal{H} = E$. The additional dimension for nonnull geodesics reveals the relation of the variable u to the Hamilton principle

$$u = \frac{1}{2}\frac{t}{a^2} - \int \mathcal{L}\, dt + b = 2\int V\, dt + 2bt \tag{10}$$

In the case of null geodesics the parameter u assumes the following form:

$$u = -\int \mathcal{L}\, dt \tag{11}$$

Formally, the case of null geodesics can be obtained if we put $a = \infty$. For a simple relativistic system $E = 0$, and, in the case of nonnull geodesics, we have

$$b = \frac{1}{2a^2}, \qquad E = 0, \qquad a = \text{const} \tag{12}$$

If $\bar{g}$ and g denote the determinants of $[g_{\alpha\beta}]$ and $[g_{ij}]$, respectively, we have

$$\bar{g} = Ag \tag{13}$$

It is worth mentioning that for simple classical mechanical systems, for which g is positive-definite, we obtain that the metric (6) has the Lorentzian signature if $V + b < 0$, and the Euclidean signature if $V + b > 0$. The signature changes if $(V + b)$ changes sign, and thus the metric (6) is singular if $V + b = 0$. We obtain, in general, the relation

$$ds^2 = 2(E + b)\, dt^2 = a^{-2}\, dt^2$$

which informs us that the timelike tangent vector to the trajectory u is unique,

$$\|u\|^2 = g_{\mu\nu} \frac{dq^\mu}{ds} \frac{dq^\nu}{ds} = 1$$

Therefore, the Eisenhart procedure reduces the problem to the study of spacelike or null geodesics.

According to general relativity, trajectories of massive particles and photons in gravitational fields are, from the very beginning, timelike or null geodesics in a spacetime with a Lorentzian metric. Here we show that also trajectories of simple mechanical systems with $V + b < 0$ can be represented as geodesics in a fictitious 'spacetime' with Lorentzian signature (the variable u is treated as time variable). Instead of studying trajectories, one can equivalently consider the problem of motion of a fictitious particle in a stationary spacetime manifold without the boundary, and conversely the problem of motion of test particles or photons in spacetime can be studied as a simple mechanical system.

2. TRAJECTORIES OF SIMPLE MECHANICAL SYSTEMS FROM THE MOTION OF A FICTITIOUS PARTICLE IN A STATIONARY BACKGROUND

Results of the previous section show that the study of a simple mechanical system can be translated into the study of a test particle and photon motion in spacetimes of general relativity. There are several advantages of doing so. The majority of problems can be reduced to problems of geodesic motions, for which the notion of differential geometry on manifolds often gives more transparent and deeper insight into the underlying symmetry. Moreover, a geodesic motion also can be formulated as a Hamiltonian system, and all the techniques of searching for integrals developed for Hamiltonian dynamics can be used to obtain integrals which do not admit an obvious geometrical interpretation.

On the metric manifold, the metric (6) (not necessary Riemannian) is written in condensed form, and then the geodesic motion is determined by the Hamiltonian

$$\mathcal{H} = \frac{1}{2m} g(P, P) = \frac{1}{2m} g^{\mu\nu} P_\mu P_\nu \tag{14}$$

which is equivalent to the geometrical equations of motion in terms of covariant derivatives.

Let us consider a particle trajectory with the momentum components P^μ in the spacetime manifold with coordinates x^ν (Greek indices run from 0 to 3, and $x^0 = u$, $x^i = q^i$). The corresponding mass parameter m is given by the condition

$$m^2 = -g_{\mu\nu} P^\mu P^\nu \tag{15}$$

where m represents the rest mass.

Henceforth we assume that the spacetime $\mathcal{M}$ with Eisenhart metric has the Lorentzian signature $(-+++)$ (the minus sign refers to the 'time' u direction). The results for the Euclidean signature can be reinterpreted in a simple way. Any trajectory $x^\mu(\lambda)$ may be conveniently regarded as an integral curve of the equations

$$m \frac{dx^\mu}{d\lambda} = P^\mu = g^{\mu\nu} P_\nu \tag{16}$$

where λ is an affine parameter along geodesics, and P^μ is determined from equation (15) as a function of x^ν.

Without any loss of generality we can consider only the case of a stationary (static) metric (6). This metric is characterized by the existence of a timelike Killing vector K^μ. Thus, it is possible to choose a frame of reference with a fictitious time coordinate

$$x^0 = u \tag{17}$$

with respect to which we have $K^\mu = \delta^\mu_0$. This means that the corresponding partial derivative of the metric is zero, i.e., $\partial g_{\mu\nu}/\partial x^0 = 0$. Thus, the vector field P^μ is stationary, i. e., $\partial P^\mu/\partial x^0 = 0$. This allows us to include directly a projected trajectory given by

$$m \frac{dx^i}{d\lambda} = P^i \tag{18}$$

into the n-dimensional quotient manifold $\mathcal{M}/\mathcal{G}$, where $\mathcal{G}$ is a group of 'time' transformations $u \to u + \Delta u$. The coordinate Killing vector $(\partial/\partial u)$, the generator of the infinitesimal group of isometry, is associated with the action of this group.

This quotient manifold has an induced positive-definite metric with components γ_{ij} which can be read out from the full $(n + 1)$-dimensional metric by decomposing it into the form

$$\begin{aligned} ds^2 &= g_{00}\, du^2 + 2g_{0i}\, du\, dx^i + \gamma_{ij}\, dx^i\, dx^j \\ &= g_{00}\, du^2 + \gamma_{ij}\, dx^i\, dx^j \\ &= g_{00}\, du^2 + dl^2 \end{aligned}$$

This is equivalent to setting

$$g_{00} = A, \qquad g_{0i} = 0, \qquad g_{ij} = \gamma_{ij}$$

Now, let us introduce a new conformally modified positive-definite metric

$$d\hat{l}^2 = \hat{g}_{ij}\, dx^i\, dx^j \tag{19}$$

on the quotient $\mathcal{M}/\mathcal{G}$ space by setting

$$d\hat{l}^2 = \frac{1 + hg_{00}}{-g_{00}}\, dl^2 \tag{20}$$

where $h =$ const. For a null geodesic, $h = 0$; for timelike geodesics, $h > 0$; and $h < 0$ for a spacelike one. Our aim is to find a one-to-one correspondence between the motion of particles or photons in the spacetime background and trajectories of a simple mechanical system. Thus, it is natural to compare the above metric with the Jacobi metric; then we obtain

$$\frac{1 + hg_{00}}{-g_{00}} = 2(E - V) \tag{21}$$

The metric (20) is a positive-definite metric on the quotient space $\mathcal{M}/\mathcal{G}$ of dimension n. The constant h in (20) is related to the proper energy defined as the total energy of the particle per mass $\underline{E} = \mathcal{E}/m$. Notice that u is a cyclic coordinate for the system with Lagrangian $\mathcal{L} = (m/2)\,(ds/d\lambda)^2$. Thus, the corresponding momentum has to be conserved

$$P_0 = \frac{\partial\mathcal{L}}{\partial(du/d\lambda)} = m\, g_{00}\, \frac{du}{d\lambda} = -m\overline{E} = -m\, \frac{1}{\sqrt{h}} \tag{22}$$

This implies the relation between h and $\overline{E}$

$$\mathscr{E} = m\overline{E} = m\frac{1}{\sqrt{h}} \to h = \overline{E}^{-2} \tag{23}$$

From (20), (21), and (9) we obtain

$$h = -a^{-2} = \overline{E}^{-2} \tag{24}$$

Let us notice that in the Eisenhart geometry we study spacelike geodesics and thus $h < 0$ or $\overline{E}$ is pure imaginary. If $a \to \infty$, i.e., for the case of null geodesics, we have $b = -E$ and $h = 0$. Relations (23) and (24) establish the one-to-one correspondence between the Jacobi geometry of simple classical dynamical systems and the geometry of fictitious particles moving in the spacetime with the Eisenhart metric.

In the special case $h = 0$, the metric (20) coincides with the so-called Fermat or optical metric. Abramowicz *et al.* (1988; Abramowicz and Prasanna, 1989) studied the role of this optical reference geometry for describing test particle trajectories in conformally projected three-space with metric (20) for $h = 0$. With such a projection in the static spacetime null lines of the four-dimensional manifold correspond to the three-dimensional space geodesics. One can easily see this fact considering Fermat's principle in its relativistic formulation (Misner *et al.*, 1973). This principle states that if $\mathcal{M} = R \times \Sigma$ is a static spacetime with the metric $g = g_{00}\, dt^2 + g_{ij}\, dx^i\, dx^j$, where Σ is a 3-manifold of constant time with Riemannian metrix ${}^{(3)}g$, and $g_{00} < 0$ is a smooth function. Neither function g_{00} nor metric ${}^{(3)}g$ depends on t. Thus, the null geodesics of $(\mathcal{M}, g)$, when projected onto Σ, are precisely the Riemannian geodesics of the 3-geometry

$$\left(\Sigma, \frac{{}^{(3)}g}{-g_{00}}\right) \tag{25}$$

and, furthermore, the affine parameter λ (i.e., the arc length) along the projected geodesics in the g metric is precisely the static time coordinate t measured along the null geodesics in $(\mathcal{M}, g)$. The above principle has a simple generalization to the case of nonnull geodesics (Szydlowski, 1996).

On the other hand, we can regard the variational principle in the reduced space

$$\delta \int d\tilde{l}^2 = 0 \Leftrightarrow \delta \int n^2(x^1, \ldots, x^n)\, dl^2 = 0 \tag{26}$$

as the variational principle in geometrical optics considering the problem of a light beam in an inhomogenous medium characterized by the refraction factor $n(x^j)$ in space with metric dl^2. Therefore, instead of studying the problem of geodesics (null, spacelike, or timelike) in the Eisenhart metric, one can equivalently investigate the problem of geodesics in a Riemannian or pseudo-Riemannian manifold with metric (20). The possibility of such a reduction appears as a consequence of 'the static form' of the spacetime metric. From the mathematical point of view, the reduced space corresponds to the conformally adjusted quotient space metric.

3. CORRESPONDENCE OF CLASSICAL MECHANICAL SYSTEMS AND GEODESIC MOTION IN SPACETIME—EXAMPLES

As the most obvious illustration, let us consider how the type of geometrical representation introduced above works in the simplest spherical example, namely that of the Schwarzschild solution, whose standard coordinate expression is (Abramowicz *et al.*, 1988; Abramowicz and Prasanna, 1989)

$$ds^2 = g_{00}\, dt^2 + \frac{1}{-g_{00}}\, dr^2 + r^2\, d\Omega^2 \tag{27}$$

with $g_{00} = -(1 - 2M/r)$. Here M is the total mass of the central spherical object, and $d\Omega$ is an infinitesimal element of solid angle.

There exist two different ways in which the space geometry can be obtained from the metric (27). Direct projection into the $t =$ const hypersurface (3D) gives the following 3-geometry:

$$dl^2 = \frac{1}{-g_{00}}\, dr^2 + r^2\, d\Omega^2 \tag{28}$$

It will be convenient to replace (28) by an equivalent form

$$dl^2 = \left(1 + \frac{M}{2\bar{r}}\right)^2 (d\bar{r}^2 + \bar{r}^2\, d\Omega^2) \tag{29}$$

where the new radial coordinate is defined by the following relation:

$$r = \left(1 + \frac{M}{2\bar{r}}\right)^2 \bar{r} = \sigma\bar{r} \tag{30}$$

Now we can look at the conformally rescaled metric (29) as the Jacobi metric of a certain simple mechanical system. Comparison of conformal factors gives the following expression for the potential function:

$$V = -\frac{1}{2}\frac{M}{\bar{r}} - \frac{1}{8}\frac{M}{\bar{r}^2} \tag{31}$$

It is worth noticing that in a direct projection, we just neglect information about the spacetime which is contained in the (00) component of the metric.

An alternative to the directly projected 3-geometry (and n-geometry in general), the model representing geodesics of spacetime as geodesics of the quotient Riemannian space (20) (Abramowicz *et al.*, 1988; Abramowicz and Prasanna, 1989), is more dynamically informative. For simplicity, let us consider the case of null geodesics in spacetime ($h = 0$); then the metric element on quotient space has the form

$$d\bar{l}^2 = \bar{\sigma}^2 \,(d\bar{r}^2 + \bar{r}^2 \, d\Omega^2) \tag{32}$$

where the new, so-called optical conformal factor $\bar{\sigma}$ is given by

$$\bar{\sigma} = \left(1 + \frac{M}{2\bar{r}}\right)^3 \left(1 - \frac{M}{2\bar{r}}\right)^{-1} \tag{33}$$

Comparing (33) with (30), we can see the basic difference in both models of spatial geometry. As mentioned in previous sections, there exists a one-to-one correspondence between the metric (27) and a simple mechanical system. Now we would like to find it. Comparing (27) with Eisenhart's metric, we obtain

$$ds^2 = -\frac{(1 - M/2\bar{r})^2}{(1 + M/2\bar{r})^2}\, dt^2 + \left(1 + \frac{M}{2\bar{r}}\right)^2 d\sigma^2 \tag{34}$$

and

$$-\frac{(1 - M/2\bar{r})^2}{(1 + M/2\bar{r}^2} = \frac{1}{2(V + b)} \tag{35}$$

where $d\sigma^2$ is a flat metric in 3D. For simplicity, we choose $b = 0$ and then the corresponding mechanical system has the following Lagrange function:

$$\mathcal{L} = \frac{1}{2}\left(1 + \frac{M}{2\bar{r}}\right)^2 \delta_{ij} \frac{dq^i}{dt}\frac{dq^j}{dt} + + \frac{(1 + M/2\bar{r})^2}{2(1 - M/2\bar{r})^2} \tag{36}$$

Thus

$$\mathcal{L} = \frac{1}{2}\left(1 + \frac{M}{2\bar{r}}\right)^2 \left\{\sigma_{ij} \frac{dq^i}{dt}\frac{dq^j}{dt} + \frac{1}{(1 - M/2\bar{r})^2}\right\} \tag{37}$$

The Lagrangian (37) can be treated as a Lagrangian of a classical system with the lapse function N (Schmidt, 1996)

$$N = \left(1 + \frac{M}{2\bar{r}}\right)^2$$

and

$$\mathcal{L} = N^2\left(\frac{1}{2}\, g_{ij}\dot{q}^i\dot{q}^j - V(q)\right)$$

where $(q^1, q^2, q^3) = (\bar{r}, \theta, \phi)$. Reparametrizing time according to the rule

$$dt = N\, d\tau$$

we obtain the Lagrangian in the form

$$\mathcal{L} = \frac{1}{2}\, \delta_{ij}\, \frac{dg^i}{d\tau}\, \frac{dq^j}{d\tau} - \tilde{V}(q)$$

where

$$\tilde{V}(q) = -\frac{1}{2}\left(\frac{2\bar{r} + M}{2\bar{r} - M}\right)^2$$

For small M/r the potential $\tilde{V}$ takes the form of the potential function for the relativistic Kepler problem (Thirring, 1977).

Now, we will go in the opposite direction: we take a simple mechanical system and we derive from it a four-dimensional geometry. As a nontrivial example we chose a rigid body with a fixed point (Arnold *et al.*, 1988). This mechanical system is important because for more than 200 years it has attracted the attention of the most prominent scientists of the epoch. The kinetic energy of the problem has the form

$$T = \frac{1}{2}\,(A\omega_1^2 + B\omega_2^2 + C\omega_3^2) \tag{38}$$

where (A, B, C) are the body's principal moments of inertia, and $(\omega_1, \omega_2, \omega_3)$ is the total angular velocity of the body in the principal axis frame. As Lagrangian coordinates we can take the classical Euler angles $(q^1, q^2, q^3) = (\phi, \theta, \psi)$. In terms of these variables the kinetic energy has the form

$$T = \frac{1}{2}\, g_{ij}\, \frac{dq^i}{dt}\, \frac{dq^j}{dt} \tag{39}$$

where

$$g_{11} = A + (C - A)\cos^2 q^2 + (B - A)\sin^2 q^2 \cos^2 q^3$$

$$g_{12} = \frac{1}{2}\,(A - B)\sin q^2 \sin 2q^3$$

$$g_{13} = C\cos q^2$$

$$g_{22} = B + (A - B)\cos^2 q^3$$
$$g_{33} = C$$

Usually the potential of the external field is assumed to have the form

$$V = V(x^1, x^2, x^3)$$

where

$$x^1 = \sin q^2 \sin q^3, \qquad x^2 = \sin q^2 \cos q^3, \qquad x^3 = \cos q^2$$

In the most important case—the heavy rigid top—the potential has the form

$$V = l_i x^i$$

where l_i are real parameters. For this choice of V there are only three points in parameter space (A, B, C, l_1, l_2, l_3) for which the Lagrangian system $\mathscr{L} = T - V$ is integrable; these are the Euler, Lagrange, and Kowalewsky cases, respectively. It seems that integrable cases of this simple mechanical system should give rise to specific symmetries of the corresponding space with the respective Eisenhart metric. In general, this problem is open. It will be interesting to identify the type of the obtained Eisenhart space-time. Results of our investigations of these problems will be published elsewhere.

4. APPLICATIONS—A NEW MODEL OF MINISUPERSPACE

There are some important reasons for studying the local instability trajectories of simple mechanical systems in terms of geodesics.

The specific property of general relativity is its gauge freedom. In particular, the property of sensitive dependence on initial conditions (hereafter SDIC for short) should be invariant with respect to time reparametrization. Additionally, as is well known, the stability in the Lyapunov sense (or boundedness) is not invariant with respect to rescaling of the time variable.

The full information about the local behavior of geodesics can be obtained from only the internal geometry, without integrating the equations of motion. Thus, this information has an invariant character from the very beginning (Szydlowski and Szczesny, 1994). The property of local instability of geodesic congruence can be expressed in terms of invariants of the internal geometry such as the Riemannian tensor and sectional curvature. If the system is locally unstable, then the nearby geodesics diverge exponentially, which implies the property of SDIC. For the Eisenhart geometry the sign of the sectional curvature can be used as a criterion of local instability (Szydlowski and Szczesny, 1994)

$$K_{u;n}(x) < 0$$

This means that the sectional curvature at any point and in any two-plane which is determined by two-directions $u \wedge v$ is negative. After averaging the sectional curvature over all two-directions the above criterion takes a very simple form (Szydlowski and Szczesny, 1994; Szydlowski, 1994)

$$R < 0$$

where R is a Ricci scalar for the Eisenhart metric. In our case, other theorems concerning the global behavior of geodesics on a smooth manifold can be adopted.

In many attempts to quantize general relativity, the space of all solutions of Einstein's equations, which are represented as geodesics, plays the fundamental role (Szydlowski and Szczesny, 1994). This space is called superspace. We will concentrate only on some subclasses of superspace, namely on the minisuperspace of the homogeneous cosmological models (for the corresponding Hamiltonian see case 9 in Table I). Misner (1972) constructed the minisuperspace using the Jacobi–Maupertuis geometry *implicitly* (Szydlowski *et al.*, 1996). Unfortunately, such a geometrical formulation of dynamical behavior faces some difficulties as a consequence of the degeneration of the Jacobi metric on the boundary of space admissible for motion (Misner, 1972). The approach based on Eisenhart's metric seems to be more attractive in this context. For vacuum cosmological models the Eisenhart metric takes the form

$$ds^2 = \frac{1}{2}\, d\bar{g}_{ij}\, d\bar{g}_{mn}\, \bar{g}^{im}\bar{g}^{jn} + \frac{a^2}{2V(a^2 + 1)}\, du^2 \tag{40}$$

where the line element of the $(n + 1)$-dimensional universe has the form

$$ds^2 = -N^2\, dt^2 + g_{ij}\, dx^i\, dx^j \tag{41}$$

and Misner's decomposition metric of the spacetime background is

$$g_{ij} \equiv e^{2\alpha}(e^{2\beta})_{ij} = e^{2\alpha}\bar{g}_{ij} \tag{42}$$

where V is a potential function in the ADM formulation of general relativity, and N is a lapse function.

To find the respective potential function we will introduce the ADM formalism (Arnowitt *et al.*, 1962). The first step is to split the $(n + 1)$-spacetime metric $g_{\mu\nu}$ into its space and time components

$$g_{\mu\nu} = \begin{pmatrix} N_i N^i - N^2 & N_i \\ N_i & g_{ij} \end{pmatrix} \tag{43}$$

where μ, ν run from 0 to n, and i, j from 1 to n. The N_i is called the shift function. It is always possible to choose $N_i = 0$, and in this gauge the line element is given by formula (41). We shall use the Einstein–Hilbert action in the standard form

$$S_{\text{ADM}} = \int \sqrt{g} NR \, dx^{(n+1)} \tag{44}$$

where $g = \det(g_{ij})$, and R is the $(n+1)$-dimensional Ricci scalar (we assume $16\pi G = c = 1$). It is convenient to rewrite the Lagrangian function in (44) in terms of metric components and their velocities as canonical variables

$$\mathcal{L}_{\text{ADM}} = \frac{1}{4N} \sqrt{g} (g^{ij} g^{kl} - g^{ik} g^{jl}) \dot{g}^{ik} \dot{g}^{jl} + NP \tag{45}$$

with P being the spatial curvature scalar calculated from g_{ij}. It will be helpful to introduce Misner's decomposition (Misner *et al.*, 1973) of the metric (44) with β_{ab} a traceless $n \times n$ matrix and $g = e^{2n\alpha}$. We assume that α is a function of time only; then, finally, the Lagrangian takes the form

$$\mathcal{L}_{\text{ADM}} = \frac{1}{4\bar{N}} \dot{\bar{g}}_{ac} \dot{\bar{g}}_{bd} \bar{g}^{ab} \bar{g}^{cd} - \frac{n(n-1)}{\bar{N}} \dot{\alpha}^2 + N e^{2n\alpha} P \tag{46}$$

where $\bar{N} = e^{-n\alpha} N$. From its variation with respect to $\bar{N}$ we get the zero–zero component of the Einstein equations, which in turn gives us the energy function in the form

$$\mathcal{H} = \frac{1}{4\bar{N}^2} \bar{g}_{ac} \bar{g}_{bd} \bar{g}^{ab} \bar{g}^{cd} - \frac{n(n-1)}{\bar{N}^2} - e^{2n\alpha} P = 0 \tag{47}$$

At this point we will fix the gauge by assuming $\bar{N} = 1$. Then we obtain the Lagrange equations with Lagrangian (46), which is formally identical to that of a point particle moving inside the potential in the form $V = e^{-2n\alpha} P$. This curvature term is a well-defined object in terms of the metric components (Landau and Lifshitz, 1971)

$$V = e^{-2n\alpha} P = e^{2(n-1)\alpha} \left\{ \frac{1}{2} \sum_{a \neq b \neq c} (C^{a}_{bc} e^{\beta^a - \beta^b - \beta^c})^2 + C_{aab} C^{a}_{bc} e^{\beta^a - 2\beta^b - \beta^c} + \sum_a D^a e^{-\beta^a} \right\} \tag{48}$$

where C^{a}_{bc} are the structure constants which define the isometry group of spacelike sections, $e^{2\beta} = \text{diag}(e^{2\beta^1}, \ldots, e^{2\beta^n})$, and $\Sigma_a \beta^a = 0$. Finally, the metric (40) with a potential function in the above form constitutes the new minisuperspace of all homogeneous cosmological models [called Bianchi models (Landau and Lifshitz, 1971)], which are now represented as geodesics in a space with one additional dimension. In formula (40) the constant b should be chosen such that relation (9) is consistent with the Hamiltonian constraint. If the potential function is neither positive nor negative, then the space with Eisenhart metric (40) can change signature. Therefore, the dynamics of minisuperspace in our sense can be represented by smoothly joining geodesics from different domains in the (q^i, u) space. The situation is similar to that which appears in the quantum cosmology where models with a changing signature are considered. A singularity in the Eisenhart metric appears if $V = -b$, but it forms a set of $(n - 1)$ dimensions in $(n + 1)$-dimensional space.

Let $E = 0$; then reduction of a dynamical system to the problem of nonnull geodesics leads to $b = 1/2a^2$, $a =$ const. Therefore, for $V \approx 0$, the Eisenhart metric takes the form of the Minkowski metric with the signature $(-+ \cdots +)$, whereas in the analogous situation the Jacobi metric $2|V|\eta$, where $\eta = \mathrm{diag}(-1, \ldots, 1)$, is degenerate on the boundary $V = 0$ and produces a singular set of dimension $(n - 1)$ in n-dimensional configuration space. Let us remark that for the Bianchi VIII model for which the potential function is nonnegative, the Eisenhart metric is singularity-free. As it was pointed out by Misner, the so-called Bianchi IX cosmological model is represented in Misner minisuperspace by a complicated trajectory which is asymptotically dominated (for $\alpha \to \infty$) by three planes forming the faces of an inverted pyramid [in α, β_+, β_- space (Misner, 1972)]. The potential function V both in the ADM Hamiltonian formalism of cosmological models, i.e., in Misner, and in our approach is the same. The disappearance of singularities in our approach (as we consider the evolution of the universe near the cosmological singularity), which has been the source of many complications (Szydlowski and Szczesny, 1994), seems to be very attractive from the point of view of future applications. There is an analogy between the Kaluza–Klein approach to the unification of the fundamental forces and our approach to the minisuperspace concept, namely, in both cases one starts from more fundamental multidimensional metrics and then physical space (dynamical trajectories in our case) is obtained via some dimensional reduction procedure.

ACKNOWLEDGMENTS

This work was supported by KBN program no. 2P03D 02211. The authors are very grateful to Prof. M. Heller for discussion and comments.

REFERENCES

Abramowicz, M. A., and Prasanna, A. R. (1989). *Monthly Notices of the Royal Astronomical Society*.

Abramowicz, M. A., Carter, B., and Lasota, J. P. (1988). *General Relativity and Gravitation*, **20**, 1173.
Arnold, V. I., Kozlov, V. V., and Neishtadt, A. I. (1988). Mathematical aspects of classical and celestial mechanics, in *Dynamical Systems III, Encyclopedia of 'Mathematical Sciences 3*, V. I. Arnold, ed., Springer-Verlag, Berlin.
Arnowitt, R., Deser, S., and Misner, C. W. (1962). The dynamics of general relativity, in *Gravitation: An Introduction to Current Research,* L. Witten, ed., Wiley, New York.
Biesiada, M., and Szydlowski, M. (1991). *Physics Letters A*, **160**, 123.
Blanco, S., Costa, A., and Rosso, O. A. (1995). *General Relativity and Gravitation*, **27**, 1295,
Bogoyavlenski, O. I. (1976). *Communications in Mathematical Physics*, **51**, 201.
Bogoyavlenski, O. I., and Novikov, S. P. (1973). *Soviet Physics-JETP*, **64**, 1475.
Calzetta, E., and El Hasi, C. (1993). *Classical and Quantum Gravity*, **10**, 1825.
Calzetta, E., and Gonzales, J. J. (1995). *Physical Review D*, **51**, 6821.
de Witt, B. (1967). *Physical Review*, **160**, 1113.
Eisenhart, L. P. (1929). *Annals of Mathematics*, **30**, 591.
Feinberg, J., and Peleg, V. (1995). *Physical Review D*, **52**, 1988.
Hawking, S. W., and Luttrell, J. C. (1984). *Nuclear Physics B*, **247**, 250.
Kamenschchik, A. Yu, Khalatnikov, I. M., and Toporensky, A. V. (1996). *Physics Letters B*, **357**, 36.
Landau, L. D., and Lifshitz, E. M. (1971). *The Classical Theory of Fields*, Pergamon Press, Oxford.
Misner, C. W. (1972). In *Magic without Magic*, J. Klauder, ed., Freeman, San Francisco.
Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation* , Freeman, San Francisco.
Schmidt, H. J. (1996). *Journal of Mathematical Physics*, **37**, 1244.
Schutz, B. (1985). *A First Course in General Relativity*, Cambridge University Press, Cambridge.
Szydlowski, M. (1994). *Journal of Mathematical Physics*, **35**, 1850.
Szydlowski, M. (1997). The generalized local instability criterion from the geodesic deviation equation, in *Geometry and Nature: In Memory of W. K. Clifford: A Conference on New Trends in Geometrical and Topological Methods in Memory of William Kingdon Clifford*, American Mathematical Society, Providence, Rhode Island.
Szydlowski, M., and Krawiec, A. (1996). *Physical Review D*, **53**, 6893.
Szydlowski, M., and Pajdosz, G. (1989). *Classical and Quantum Gravity*, **6**, 1391.
Szydlowski, M., and Szczesny, J. (1994). *Physical Review D*, **50**, 819.
Szydlowski, M., Heller, M., and Sasin, W. (1996). *Journal of Mathematical Physics*, **37**, 346.
Thirring, W. (1977). *Klassiche Dynamische Systeme*, Springer-Verlag, Berlin.
Varvoglis, H., and Papadopolous, D. (1992). *Astronomy and Astrophysics*, **261**, 664.